Yang Zhang

(Dated: September 23, 2019)

## I. NOTATIONS

Spinor helicity is a very useful formalism to present both massless and massive kinematics. In this formalism, the polarization vectors have a particularly simple form. The famous BCFW recursion relation also becomes manifest. Spinor helicity is a basis language of modern scattering amplitude.

We start from $4D$ massless (null) vectors' spinor helicity formalism. Since any massive vector is a linear combination of two massless vectors, this formalism is also useful for massive kinematics.

Spinor helicity formalism is complicated for $D > 4$.

Through this course, we use

$$(+, -, -, -) \tag{1}$$

for the Minkowski spacetime signature (West coast convention, same as that in Peskin-Schroeder's QFT book).

For a null vector $p_\mu = (p_0, p_1, p_2, p_3)$, $p^2 = 0$, we derive the spinor helicity representation. The four-component Pauli matrices are defined as,

$$\sigma^0_{\alpha\dot\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma^1_{\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^2_{\alpha\dot\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma^3_{\alpha\dot\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

These four matrices transfer under the Lorentz group as a four-vector.

The contraction

$$p_{\alpha\dot\beta} \equiv p_\mu \sigma^\mu_{\alpha\dot\beta} = \begin{pmatrix} p^0 - p^3 & -(p^1 - ip^2) \\ -(p^1 + ip^2) & p^0 + p^3 \end{pmatrix} = \begin{pmatrix} p_- & -p^\perp_- \\ -p^\perp_+ & p_+ \end{pmatrix}. \tag{3}$$

where

$$p_\pm = p_0 \pm p_3, \quad p^\perp_\pm = p_1 \pm ip_2 \tag{4}$$

**Question** Prove that $\det(p_{\alpha\dot\beta}) = 0$ if $p^2 = 0$.

Since $\det(p_{\alpha\dot\beta}) = 0$ for a null vector, from basic linear algebra

$$p_{\alpha\dot\beta} = \lambda_\alpha \tilde\lambda_{\dot\beta} \tag{5}$$

where $\lambda_\alpha$ and $\tilde{\lambda}_{\dot{\beta}}$ are holomorphic and anti-holomorphic *Weyl* spinors.

Although the explicit component form of $\lambda_\alpha$ and $\tilde{\lambda}_{\dot{\beta}}$ are not needed in most cases, we give the following explicit form,

$$\lambda_\alpha \equiv \begin{pmatrix} -zp_-^\perp/\sqrt{p_+} \\ z\sqrt{p_+} \end{pmatrix}, \quad \tilde{\lambda}_{\dot{\beta}} \equiv \begin{pmatrix} -p_+^\perp/(z\sqrt{p_+}) & \frac{\sqrt{p_+}}{z} \end{pmatrix}, \tag{6}$$

where $z$ is a unfixed parameter since this decomposition is not unique. In other words, we have the freedom,

$$\lambda_\alpha \to t\lambda_\alpha, \quad \tilde{\lambda}_{\dot{\beta}} \to t^{-1}\tilde{\lambda}_{\dot{\beta}}, \tag{7}$$

This is just the little group action on the null vector $p$. Based on this symmetry, we see that the value of $z$ is not fixed.

However, in some cases, we still want to fix $z$ for convenience. For example, when $p$ is a real null momentum and $p_+ > 0$, we can set $z$ as a complex phase such that $|z| = 1$. In this case,

$$\lambda_i = (\tilde{\lambda}_i)^*, \quad i = 1, 2 \tag{8}$$

where $*$ stands for the complex conjugate. However, since we are interested by the holomorphic properties of amplitudes, we usually avoid this choice.

Another choice is to set $z = \sqrt{p_+}$. By this choice, the apparent square root disappeared.

As usual, we use anti-symmetric matrix

$$\sigma^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{9}$$

to raise and lower the spinor indices,

$$\lambda^\alpha = \epsilon^{\alpha\beta}\lambda_\beta, \quad \lambda_\alpha = \epsilon_{\alpha\beta}\lambda^\beta, \quad \lambda^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\lambda_{\dot{\beta}}, \quad \lambda_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\lambda^{\dot{\beta}}\,. \tag{10}$$

## II. SPINOR PRODUCTS

In the section, we consider a list of null vectors $p_i$, $i = 1, \ldots$. For the vector $p_i$, we denote $\lambda(p_i)$ as $\lambda_i$.

For two null vector $p_i$ and $p_j$, the spinor products are defined as

$$\langle ij\rangle \equiv \lambda_i^\alpha \lambda_{j,\alpha} \tag{11}$$

$$[ij] \equiv \tilde{\lambda}_{i,\dot{\alpha}}\tilde{\lambda}_j^{\dot{\alpha}} \tag{12}$$

Note that this definition is sensitive to the order of the spinors.

$$\langle ij\rangle \equiv \epsilon^{\alpha\beta}\lambda_{i,\beta}\lambda_{j,\alpha} = \epsilon^{\beta\alpha}\lambda_{i,\alpha}\lambda_{j,\beta} = \epsilon^{\beta\alpha}\lambda_{j,\beta}\lambda_{i,\alpha} = -\epsilon^{\alpha\beta}\lambda_{j,\beta}\lambda_{i,\alpha} = -\langle ji\rangle\,. \tag{13}$$

Similarly, $[ij] = -[ji]$. It is clearly that $\langle ij\rangle = 0$ and $[ij] = 0$.

Note that because of the little group action in (7), neither $\langle ij\rangle$ nor $[ij]$ is uniquely defined.

For further discussion, we need to introduce the $\bar{\sigma}$ Pauli matrices,

$$\bar{\sigma}^{0,\dot{\alpha}\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \bar{\sigma}^{1,\dot{\alpha}\beta} = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \bar{\sigma}^{2,\dot{\alpha}\beta} = -\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \bar{\sigma}^{3,\dot{\alpha}\beta} = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{14}$$

These matrices are related to the original Pauli matrices by,

$$\bar{\sigma}^{\mu,\dot{\beta}\delta} = \epsilon^{\delta\alpha}\epsilon^{\dot{\beta}\dot{\alpha}}\sigma^{\mu}_{\alpha\dot{\alpha}}\,. \tag{15}$$

When there is no confusion, we may drop the spinor indices, and simply rewrite the above formula as

$$\epsilon\sigma^{\mu}\epsilon^{T} = (\bar{\sigma}^{\mu})^{T} \tag{16}$$

where $\epsilon = i\sigma_2$ is the $2\times 2$ matrix $\epsilon^{\alpha\beta}$. We have the Pauli matrix orthonormal relation,

$$\mathrm{Tr}(\sigma^{\mu}\bar{\sigma}^{\nu}) = \mathrm{Tr}(\bar{\sigma}^{\mu}\sigma^{\nu}) = 2\eta^{\mu\nu}, \quad \sigma_{\mu,\alpha\dot{\alpha}}\bar{\sigma}^{\mu,\dot{\beta}\beta} = 2\delta^{\beta}_{\alpha}\delta^{\dot{\beta}}_{\dot{\alpha}} \tag{17}$$

One application is that we can use the orthonormal relation to recover the original null vector $p_i$ from the spinors. (Here $\lambda$ stands for $\lambda_\alpha$, and $\tilde{\lambda}$ stands for $\tilde{\lambda}_{\dot{\beta}}$,)

$$p_i^{\nu} = \frac{1}{2}(\tilde{\lambda}_{i,\dot{\beta}}\bar{\sigma}^{\nu,\dot{\beta}\alpha}\lambda_{i,\alpha}) = \frac{1}{2}(\tilde{\lambda}_i\bar{\sigma}^{\nu}\lambda_i)\,. \tag{18}$$

Similarly, we may construct new null vectors as,

$$(\lambda_i\tilde{\lambda}_j)^{\nu} = \frac{1}{2}(\tilde{\lambda}_{j,\dot{\beta}}\bar{\sigma}^{\nu,\dot{\beta}\alpha}\lambda_{i,\alpha}) = \frac{1}{2}(\tilde{\lambda}_j\bar{\sigma}^{\nu}\lambda_i)\,. \tag{19}$$

It is clear that $(\lambda_i\tilde{\lambda}_j)_{\nu}\sigma^{\nu}_{\alpha\dot{\beta}} = \lambda_{i,\alpha}\tilde{\lambda}_{j,\dot{\beta}}$. Note that $p_i^{\nu} = (\lambda_i\tilde{\lambda}_i)^{\nu}$.

One important identity is that

$$2(\lambda_i\tilde{\lambda}_j)^{\mu}(\lambda_k\tilde{\lambda}_l)_{\mu} = \langle ik\rangle[lj] \tag{20}$$

It is easy to prove this identity by the standard Pauli matrix identities.

$$\langle ik\rangle[lj] = (\epsilon^{\alpha\beta}\lambda_{i,\beta}\lambda_{k,\alpha})(\epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_{l,\dot{\alpha}}\tilde{\lambda}_{j,\dot{\beta}}) \tag{21}$$

$$= (\lambda_k^{T}\epsilon\lambda_i)(\tilde{\lambda}_l\epsilon\tilde{\lambda}_j^{T}) = (\lambda_k^{T}\epsilon\lambda_i)(\tilde{\lambda}_j\epsilon^{T}\tilde{\lambda}_l^{T}) \tag{22}$$

$$= (\lambda_i\tilde{\lambda}_j)_{\mu}(\lambda_k^{T}(\bar{\sigma}^{\mu})^{T}\tilde{\lambda}_l^{T}) = (\lambda_i\tilde{\lambda}_j)_{\mu}(\tilde{\lambda}_l\bar{\sigma}^{\mu}\lambda_k) \tag{23}$$

$$= (\lambda_i\tilde{\lambda}_j)_{\mu}(\lambda_k\tilde{\lambda}_l)_{\nu}\,\mathrm{Tr}(\sigma^{\nu}\bar{\sigma}^{\mu}) = 2(\lambda_i\tilde{\lambda}_j)^{\mu}(\lambda_k\tilde{\lambda}_l)_{\mu}\,. \tag{24}$$

## III. IDENTITIES

- Schouten Identity.

$$\langle ij\rangle\langle kl\rangle + \langle jk\rangle\langle il\rangle + \langle ki\rangle\langle jl\rangle = 0 \tag{25}$$

It is easy to prove this identity by a simple expansion.

$$\lambda_i = c_1\lambda_j + c_2\lambda_k \tag{26}$$

From the spinor products, we have

$$\langle ij\rangle = c_2\langle kj\rangle, \quad \langle ik\rangle = c_1\langle jk\rangle, \tag{27}$$

Hence

$$\langle il\rangle = c_1\langle jl\rangle + c_2\langle kl\rangle = \frac{\langle ik\rangle}{\langle jk\rangle}\langle jl\rangle + \frac{\langle ij\rangle}{\langle kj\rangle}\langle kl\rangle \tag{28}$$

This is a very useful identity.

Note that since Schouten identity is a quadratic relation for $\langle\rangle$ and $[]$, it is not clearly how to eliminate dependent spinor products by naively using this identity.

- Momentum Conservation. Suppose that we have a scattering process of $n$ massless particles. $p_i^2 = 0$ and $\sum p_i =$, $i = 1, \ldots n$. Then we have

$$\sum_{i=1}^{n}\langle ik\rangle[ji] = 0, \quad ,\forall k, j \tag{29}$$

Note that the identity contains $n-1$ terms if $k = j$, or $n-2$ terms if $k \neq j$.

Again this is a quadratic relation for $\langle\rangle$ and $[]$, it is not straightforward to use this identity.

- For four null vectors $p_i$, $p_j$, $p_k$ and $p_l$, we have

$$\langle ij\rangle[jk]\langle kl\rangle[li] = 2(p_i\cdot p_l)(p_j\cdot p_k) + 2(p_i\cdot p_j)(p_k\cdot p_l) - 2(p_i\cdot p_k)(p_j\cdot p_l) - 2i\epsilon(i,j,k,l) \tag{30}$$

where $\epsilon(i,j,k,l) = \epsilon^{\mu_1\mu_2\mu_3\mu_4}p_{i\mu_1}p_{j\mu_2}p_{k\mu_3}p_{l\mu_4}$. $\epsilon^{\mu_1\mu_2\mu_3\mu_4}$ is the totally antisymmetric tensor. This identity is from the Pauli matrix identities.

$$\mathrm{Tr}(\sigma^\mu\bar\sigma^\nu\sigma^\tau\bar\sigma^\rho) = 2(\eta^{\mu\nu}\eta^{\tau\rho} + \eta^{\mu\rho}\eta^{\nu\rho} - \eta^{\mu\tau}\eta^{\nu\rho} + i\epsilon^{\mu\nu\tau\rho}) \tag{31}$$

$\epsilon(i,j,k,l)$ appears in amplitudes frequently. It is Lorentz invariant but partiy-odd.

We may further define that for any four vectors $\{e_1, e_2, e_3, e_4\}$, a $4\times 4$ matrix $G(e_1, e_2, e_3, e_4)$,

$$G(e_1, e_2, e_3, e_4)_{ij} \equiv e_i \cdot e_j \,. \tag{32}$$

Its determinant is $g(e_1, e_2, e_3, e_4) = \det G(e_1, e_2, e_3, e_4)$. $G$ is called the "Gram matrix" of $\{e_1, e_2, e_3, e_4\}$. $g(e_1, e_2, e_3, e_4)$ is clearly Lorentz invariant and partiy-even. Furthermore, we have,

$$g(e_1, e_2, e_3, e_4) = -\epsilon(i, j, k, l)^2 \,. \tag{33}$$

## IV. APPLICATIONS

Polarization. Consider a gluon or photon with the momentum $p_i$, $p_i^2 = 0$. One big advantage of spinor helicity formalism is to construct the polarization vector, with specific helicity.

$$\epsilon^\mu_{i,+} = \sqrt{2}\frac{(\lambda_k \tilde{\lambda}_i)^\mu}{\langle ki\rangle}, \quad \epsilon^\mu_{i,-} = \sqrt{2}\frac{(\lambda_i \tilde{\lambda}_k)^\mu}{[ik]} \tag{34}$$

Then it is easy to check that

$$p_i \cdot \epsilon_{i,\pm} = 0, \quad \epsilon^2_{i,\pm} = 0, \quad \epsilon^\mu_{i,+}\epsilon_{i,-\mu} = -1 \tag{35}$$

The completeness relation reads,

$$\epsilon^\mu_{i,+}\epsilon^\nu_{i,-} + \epsilon^\mu_{i,-}\epsilon^\nu_{i,+} = -\eta^{\mu\nu} + \frac{p_i^\mu p_k^\nu + p_i^\nu p_k^\mu}{p_i \cdot p_k} \tag{36}$$

**Exercise** For the four-point massless kinematics, try to convert

$$\frac{\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle} \tag{37}$$

to a function of Mandelstam variables.

**Solution** We use the spinor product formula (30).

$$\frac{\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle} = \frac{\langle 12\rangle\langle 34\rangle[31][42]}{s_{13}s_{24}} = \frac{\langle 21\rangle[13]\langle 34\rangle[42]}{s_{13}s_{24}} \tag{38}$$

$$= \frac{s_{12}s_{34}/2 + s_{24}s_{13}/2 - s_{23}s_{14}/2}{s_{13}s_{24}} = \frac{s^2/2 - t^2/2 + (s+t)^2/2}{(s+t)^2} \tag{39}$$

$$= \frac{s}{s+t} \tag{40}$$

We see this computation is not straightforward. For more complicated spinor functions, the simplification will be more involved. Hence we introduce the modern method, "momentum twistor" for these computations.